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Black Hole and Cosmic Entropy for Schwarzschild–de Sitter Space–Time

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We calculate the free energy and the entropy of a scalar field in terms of the brick-wall method in the background of the Schwarzschild–de Sitter space–time. We obtain the entropy of a black hole and the cosmic entropy at nonasymptotic flat space. When the cut-off satisfies the proper condition, the entropy of a black hole is proportional to the area of a black hole horizon, and the cosmic entropy is proportional to the cosmic horizon area.

A significant development in physics over the past 30 years is the research of black hole physics. The area of the event horizon of a black hole can be interpreted as entropy and the surface gravity can be looked as temperature (Howking, 1972; Bekenstein *et al.*, 1973). The four laws of a black hole thermodynamics is found (Bekenstein, 1972; Bekenstein *et al.*, 1973; Smarr, 1973). Hawking's discovery of the thermal radiation of a black hole supported the idea that a black hole has temperature. For the last few years, the research of the black hole temperature has achieved high perfection. However, the study of the black hole entropy has not been satisfactory (Frolov and Page, 1993; Unruh and Wald, 1982). Recently, one of the most intriguing problems in black hole physics has been the study of its entropy (Cognola and Lecca, 1998; Lee *et al.*, 1996; Solodukhin, 1995a,b; Shen and Chen, 1999). The brick-wall method has been generally used ('t Hooft, 1985). 't Hooft examined the statistical property of a free scalar field in the background of the Schwarzschild black hole by using the brick-wall method and obtained an expression of entropy in terms of the area of the horizon. Furthermore, 't Hooft proved that entropy is proportional to its horizon area. When cut-off satisfies the proper condition, entropy can be written as $S = A_{\mathrm{H}}/4$. When cut-off approaches zero, entropy will diverge. He proposed that the divergence was caused by the infinite density of states near horizon. Afterward, the entropy of every black hole

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was studied in an asymptotic flat space by this method (Cognola and Lecca, 1998; Lee *et al.*, 1996; Shen and Chen, 1999). But in a nonasymptotic flat space, the entropy for a black hole still remains, which needs to be solved.

In this paper, we calculate the entropy and the free energy of a free scalar field near the horizon of a black hole and the universe by using the brick-wall method in the Schwarzschild–de Sitter space–time. We obtained a relation between entropy and horizon area: when the cut-off satisfies the proper condition relation between entropy and horizon of the black hole, $S_{\mathrm{H}} = A_{\mathrm{H}}/4$, and the relation between cosmic entropy and area of the horizon, $S_{\mathrm{c}} = A_{\mathrm{c}}/4$. When cosmic factor $\Lambda \to 0$, we obtain the known result ('t Hooft, 1985). In the Boyer–Lindquist coordinate system, the metric of a Schwarzschild–de Sitter black hole is given by:

$$\begin{aligned} ds^2 = & -(1 - 2Mr^{-1} - \Lambda r^2 3^{-1})\, dt^2 + (1 - 2Mr^{-1} - \Lambda r^2 3^{-1})^{-1}\, dr^2 \\ & + r^2(d\theta^2 + \sin^2\theta \ \ d\varphi^2). \end{aligned} \tag{1}$$

Let $\Lambda > 0$ and $9\Lambda M^2 < 1$, then, the horizon of black hole and the cosmic horizon are

$$\begin{aligned} r_+ &= \frac{2}{\sqrt{\Lambda}} \cos\left(\alpha + \frac{\pi}{3}\right), \\ r_{++} &= \frac{2}{\sqrt{\Lambda}} \cos\left(\alpha - \frac{\pi}{3}\right), \quad \text{and} \\ \alpha &= \frac{1}{3} \arccos(3M\Lambda^{1/2}). \end{aligned} \tag{2}$$

The surface gravity of a black hole and universe are

$$\kappa_{\mathrm{H}} = \Lambda 6^{-1} r_+^{-1}(r_{++} - r_+)(r_+ - r_{--}) \quad \text{and} \tag{3}$$

$$\kappa_{\mathrm{c}} = \Lambda 6^{-1} r_{++}^{-1}(r_{++} - r_+)(r_{++} - r_{--}), \tag{4}$$

where

$$r_{--} = -\frac{2}{\sqrt{\Lambda}} \cos\alpha$$

is the negative solution of $3r - 6M - \Lambda r^3 = 0$. The entropy of a black hole and universe are

$$S_{\mathrm{H}} = \frac{1}{4} A_{\mathrm{H}} = \pi r_+^2 \quad \text{and} \tag{5}$$

$$S_{\mathrm{c}} = \frac{1}{4} A_{\mathrm{c}} = \pi r_{++}^2. \tag{6}$$

In curved space–time, the massless scalar field yield,

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\,\partial_\nu\psi\right)=0. \tag{7}$$

Using the brick-wall method, we can assume that the wave function yields ('t Hooft, 1985)

$$\psi_{r=r_{+}+\epsilon}=\psi_{r=L}=0, \tag{8}$$

where $r_{+}\ll L\ll r_{++}\quad \epsilon\ll r_{+}$.

ψ of Eq. (7) can be written as

$$\psi=e^{-iEt}\frac{R(r)}{r}Y_{l,m}(\theta,\varphi), \tag{9}$$

and the radial part of Eq. (7) can be written as

$$\begin{aligned}A^2\frac{d^2R}{dr^2}+\left(\frac{2M}{r^2}+\frac{2}{3}\Lambda r\right)A\frac{dR}{dr}+E^2R\\ -A\left(\frac{2M}{r^3}+\frac{2}{3}\Lambda-\frac{l(l+1)}{r^2}\right)R=0,\end{aligned} \tag{10}$$

where

$$A=1-\frac{2M}{r}-\frac{1}{3}\Lambda r^2.$$

By the tortoise-type coordinate transformation

$$dr_*=A^{-1}\,dr, \tag{11}$$

the Eq. (10) can be reduced to

$$\left(\frac{d^2}{dr_*^2}+E^2-V_l(r_*)\right)R(r_*)=0, \tag{12}$$

where

$$V_l=\left(\frac{2M}{r^3}+\frac{2}{3}\Lambda-\frac{l(l+1)}{r^2}\right)A.$$

By using WKB approximation, we obtained the solution of Eq. (12):

$$R(r_*)=e^{iS(r_*)}. \tag{13}$$

From $S'(r_*)=K_l(r_*)=\sqrt{E^2-V_l(r_*)}$ and the boundary condition of Eq. (8), we have

$$\int K_l(r_*)\,dr_*=\int_{r_{+}+\epsilon}^{L}K_l(r)A^{-1}\,dr=n\pi. \tag{14}$$

The temperature of black hole and universe are (Gibbons and Hawking, 1977)

$$T_{\mathrm{H}} = \frac{\kappa_{\mathrm{H}}}{2\pi}, \qquad T_{\mathrm{c}} = \frac{\kappa_{\mathrm{c}}}{2\pi}. \tag{15}$$

In terms of the theory of canonical ensemble, the free energy of a Bose system can be written as

$$\beta F = \sum_{E} \ln(1 - e^{-\beta E}). \tag{16}$$

When we look at it as semiclassic and take the energy state as a continuous distribution, we can replace by integration the sum

$$\sum_{E} \to \int_{0}^{\infty} dE\, g(E),$$

where $g(E)$ is density of states, $g(E) = d\Gamma(E)/dE$; $\Gamma(E)$ is the microstate number, which is

$$\begin{aligned}\Gamma(E) &= \sum_{l,m} n_r(E, l, m) = \sum_{l} (2l+1) n_l(E, l) \\ &\approx \int_{l} (2l+1)\; dl \frac{1}{\pi} \int K_r(E, l) A^{-1}\, dr.\end{aligned} \tag{17}$$

Thus

$$\begin{aligned}\beta F &\approx \int_{0}^{\infty} dE\, g(E) \ln(1 - e^{-\beta E}) \\ &= -\beta \int dE\, \frac{\Gamma(E)}{e^{\beta E} - 1} \\ &= \frac{2}{3} \frac{\beta}{\pi} \int \frac{dE}{e^{\beta E} - 1} \int_{r_{+}+\epsilon}^{L} dr\, A^{-2} r^{2} \left[E^{2} - \left(\frac{2M}{r^{3}} + \frac{2}{3}\Lambda \right) A \right]^{3/2} \\ &\approx \frac{2}{3} \frac{\beta}{\pi} \int \frac{E^{3}\; dE}{e^{\beta E} - 1} \int_{r_{+}+\epsilon}^{L} A^{-2} r^{2}\; dr = \frac{2\pi^{3}}{45\beta^{3}} \int_{r_{+}+\epsilon}^{L} A^{-2} r^{2}\; dr.\end{aligned} \tag{18}$$

In Eq. (18), considering $2M/r^{3} \to 0$, when $r \gg 2M$ and Λ, which is a constant of a very small value, we have

$$\int A^{-2} r^{2}\; dr = f_1(r) + f_2(r) + f_3(r), \tag{19}$$

where

$$f_1(r) = -\frac{1}{a^2}\left[-\frac{r_+^4}{(r-r_+)(r_+-r_{++})^2(r_+-r_{--})^2} + \frac{2r_+^3}{(r_+-r_{++})^2(r_+-r_{--})^2}\right.$$
$$\left.\times\left(2-\frac{r_+}{r_+-r_{++}}-\frac{r_+}{r_+-r_{--}}\right)\ln(r-r_+)\right], \tag{20}$$

$$f_2(r) = -\frac{1}{a^2}\left[\frac{r_{++}^4}{(r-r_{++})(r_+-r_{++})^2\,(r_+-r_{--})^2} + \frac{2r_{++}^3}{(r_+-r_{++})^2(r_{++}-r_{--})^2}\right.$$
$$\left.\times\left(2-\frac{r_{++}}{r_{++}-r_+}-\frac{r_{++}}{r_{++}-r_{--}}\right)\ln(r-r_{++})\right], \tag{21}$$

$$f_3(r) = -\frac{1}{a^2}\left[\frac{r_{--}^4}{(r-r_{--})(r_{--}-r_+)^2(r_{--}-r_{++})^2} + \frac{2r_{--}^3}{(r_{--}-r_+)^2(r_{--}-r_{++})^2}\right.$$
$$\left.\times\left(2-\frac{r_{--}}{r_{--}-r_+}-\frac{r_{--}}{r_{--}-r_{++}}\right)\ln(r-r_{--})\right]. \tag{22}$$

When $r_+ < r < r_{++}$, $f_2(r)$ and $f_3(r)$ can be neglected, since it may be small compared to $f_1(r)$. We only discuss $f_1(r)$.

For the definite integral, we have

$$\begin{aligned} f_1(r) \approx &-\frac{1^2}{a}\left[-\frac{r_+^4}{\epsilon(r_+-r_{++})^2(r_+-r_{--})^2} - \frac{2r_+^3}{(r_+-r_{++})^2(r_+-r_{--})^2}\right.\\ &\left.\times\left(2-\frac{r_+}{r_+-r_{++}}-\frac{r_+}{r_+-r_{--}}\right)\ln\epsilon\right]\\ &-\frac{1^2}{a}\left[-\frac{r_+^4}{(L-r_+)(r_+-r_{++})^2(r_+-r_{--})^2} + \frac{2r_+^3}{(r_+-r_{++})^2(r_+-r_{--})^2}\right.\\ &\left.\times\left(2-\frac{r_+}{r_+-r_{++}}-\frac{r_+}{r_+-r_{--}}\right)\ln(L-r_+)\right], \end{aligned} \tag{23}$$

were $a = \frac{1}{3}\Lambda$. In the right hand side of Eq. (23), the first part of every term is an intrinsic contribution from the horizon and it diverges linearly as $\epsilon \to 0$. The second part is the usual contribution from the vacuum surrounding the system at large distances and is of little relevance here. The free energy of a scalar field in the background of the Schwarzschild–de Sitter black hole in the approximation is

$$F \approx -\frac{2\pi^3 r_+^4}{45\beta^4 a^2 \epsilon (r_+ - r_{++})^2 (r_+ - r_{--})^2} + \frac{4\pi^3 r_+^3}{45\beta^4 a^2 (r_+ - r_{++})^2 (r_+ - r_{--})^2} \times \left(2 - \frac{r_+}{r_+ - r_{++}} - \frac{r_+}{r_+ - r_{--}}\right) \ln \epsilon. \tag{24}$$

Using the relation between the entropy and free energy, we have

$$S = \beta^2 \frac{\partial F}{\partial \beta} \tag{25}$$

and

$$S_{\mathrm{H}} = \frac{8\pi^3 r_+^4}{45\beta^3 a^2 \epsilon (r_+ - r_{++})^2 (r_+ - r_{--})^2} + \frac{16\pi^3 r_+^3}{45\beta^3 a^2 (r_+ - r_{++})^2 (r_+ - r_{--})^2} \times \left(2 - \frac{r_+}{r_+ - r_{++}} - \frac{r_+}{r_+ - r_{--}}\right) \ln \frac{1}{\epsilon}. \tag{26}$$

From

$$\beta = \frac{1}{T_{\mathrm{H}}} = \frac{2\pi}{\kappa} = \frac{12\pi r_+}{\Lambda (r_{++} - r_+)(r_+ - r_{--})},$$

$a = \frac{1}{3}\Lambda$ and $r_+ + r_{++} + r_{--} = 0$, we obtain

$$S_{\mathrm{H}} = \frac{A_{\mathrm{H}}}{360\beta\epsilon} + \frac{\Lambda}{540}\left(r_+^2 - 2r_+ r_{--}\right) \ln \frac{1}{\epsilon}, \tag{27}$$

where $A_{\mathrm{H}} = 4\pi r_+^2$ is the area of the horizon of a black hole. When ϵ satisfies the following relational expression

$$\epsilon = \frac{\Lambda (r_{++} - r_+)(r_+ - r_{--})}{90\pi\, 12 r_+}, \tag{28}$$

we have

$$S = \frac{1}{4} A_{\mathrm{H}} + \frac{\Lambda}{540}\left(r_+^2 - 2r_+ r_{--}\right) \ln \frac{1}{\epsilon}. \tag{29}$$

When $\Lambda \to 0$, Eq. (27) can be reduced to

$$S_{\Lambda \to 0} = \frac{A_{\mathrm{H}}}{360\beta\epsilon}.$$

In this case, if we let $\epsilon = 1/720\pi M$ [13], we get

$$S = \frac{1}{4} A_{\mathrm{H}}. \tag{30}$$

It returns the result of the Schwarzschild black hole.

The free energy of cosmic horizon can be written as

$$\beta F = \sum_E \ln(1 - e^{-\beta E}) \approx \frac{2\pi^3}{45\beta^3} \int_{r_{++}+h}^{H} a^{-2} r^2 \, dr. \tag{31}$$

where $h \ll r_{++}$, $H \gg r_{++}$. We only take $f_2(r)$ of integral expression and obtain the following expression by a similar calculation:

$$F \approx -\frac{2\pi^3 r_{++}^4}{45\beta^4 a^2 h (r_{++} - r_+)^2 (r_{++} - r_{--})^2} + \frac{4\pi^3 r_{++}^3}{45\beta^4 a^2 (r_{++} - r_+)^2 (r_{++} - r_{--})^2} \times \left(2 - \frac{r_{++}}{r_{++} - r_+} - \frac{r_{++}}{r_{++} - r_{--}}\right) \ln h. \tag{32}$$

Using the relation between the entropy and free energy

$$S = \beta^2 \frac{\partial F}{\partial \beta}, \tag{33}$$

we have

$$S = \frac{8\pi^3 r_{++}^4}{45\beta^3 a^2 h (r_{++} - r_+)^2 (r_{++} - r_{--})^2} + \frac{16\pi^3 r_{++}^4}{45\beta^3 a^2 (r_{++} - r_+)^2 (r_{++} - r_{--})^2} \times \left(2 - \frac{r_{++}}{r_{++} - r_+} - \frac{r_{++}}{r_{++} - r_{--}}\right) \ln \frac{1}{h}. \tag{34}$$

Using

$$\beta = \frac{1}{T_c} = \frac{2\pi}{\kappa_c} = \frac{12\pi r_{++}}{\Lambda (r_{++} - r_+)(r_{++} - r_{--})}$$

and $a = \Lambda 1/3$, we have

$$S = \frac{A_c}{360\beta h} + \frac{\Lambda}{540}\left(r_{++}^2 - 2r_{++} r_{--}\right) \ln \frac{1}{h}. \tag{35}$$

If h satisfies the following relational expression

$$h = \frac{\Lambda (r_{++} - r_+)(r_{++} - r_{--})}{90\pi 12 r_{++}}, \tag{36}$$

the cosmic entropy

$$S = \frac{1}{4} A_c + \frac{\Lambda}{540}\left(r_{++}^2 - 2r_{++} r_{--}\right) \ln \frac{1}{h}. \tag{37}$$

In this paper, we start with K–G equation in the background of the Schwarzschild–de Sitter black hole and calculate the free energy and entropy of a scalar field near the horizon of a black hole and universe using WKB approximation and brick-wall method. We find that the entropy of a scalar field is proportional to the area of the black hole horizon of a black hole near the horizon of a black hole and it is proportional to the area of the cosmic horizon near the cosmic horizon.

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REFERENCES

Bekenstein, J. (1972). PHD Thesis, Princeton University.
Bekenstein, J. D., Carter, B., and Hawking, S. W. (1973) *Communications in Mathematical Physics* **31**, 161.
Cognola, G. and Lecca, P. (1998). *Physical Review D* **57**, 1108.
Frolov, V. P. and Page, D. N. (1993). *Physical Review Letters* **71**, 3902.
Gibbons, G. W. and Hawking, S. W. (1977). *Physical Review D* **15**, 130.
Hawking, S. W. (1972). *Communications in Mathematical Physics* **25**, 152.
Hawking, S. W. (1975). *Communications in Mathematical Physics* **43**, 199.
’t Hooft, G. (1985). *Nuclear Physics B* **256**, 727.
Lee, H., Kim, S. W., and Kim, W. T. (1996). *Physical Review D* **54**, 6559.
Shen, Y. G. and Chen, D. M. (1999). *General Relativity and Gravitation* **31**, 315.
Smarr, L. (1973). *Physical Review Letters* **30**, 71.
Solodukhin, S. N. (1995a). *Physical Review D* **51a**, 609.
Solodukhin, S. N. (1995b). *Physical Review D* **51b**, 618.
Unruh, W. G. and Wald, R. M. (1982). *Physical Review D* **25**, 942.